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A remark on T-duality and quantum volumes of zero-brane moduli spaces

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Abstract: T-duality (Fourier-Mukai duality) and properties of classical instanton moduli spaces can be used to deduce some properties of α'-corrected moduli spaces of branes for type-IIA string theory compactified on $K3$ or $\mathbb{T}^4$. Some interesting differences between the two compactifications are exhibited.

Keywords: Superstrings and Heterotic Strings, D-branes, String Duality.

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Contents

1. Introduction

The properties of space-time in string theory are very mysterious, since space-time often arises as a derived object from more primitive concepts, e.g. worldsheet string theory, or the moduli space of vacua of a gauge theory. As such it manifests properties quite different from expectations based on ordinary classical geometry. Such properties include T-duality where a theory defined on a large circle has the same physics as a theory defined on a small circle, and space-time non-commutativity where the theory shows evidence of non-commuting coordinates. In this note, we explore some aspects of large-small dualities and observe their consequences for moduli spaces of zero branes on $K3$ and $\mathbb{T}^4$. We work with units where $4\pi^2\alpha' = 1$, so that T-duality takes $V \to 1/V$.

2. Fourier-Mukai and quantum volumes of brane moduli spaces

There is a T-duality symmetry in the $O(4,4;\mathbb{Z})$ T-duality group of type-IIA on $\mathbb{T}4$ which inverts the volume of the 4-torus, in the absence of B-fields. There is also such a symmetry in $O(4,20;\mathbb{Z})$, the duality group of a $K3$. Let us recall the set-up which is used to describe such a duality.

The moduli space of positive 4-planes in a lorentzian space $H^*(K3,\mathbb{R}) = \mathbb{R}^{(4,20)}$ describes the moduli space of compactifications of type-IIA on $K3$ [1]. Let us label basis vectors spanning the 4-plane as E^1 to E^4. They can be expressed in terms of the moduli as

$$E^1 = \left(V - \frac{1}{2}B.B, 1; 0\right), \qquad E^i = (0, -B.\omega^i; \omega^i). \tag{2.1}$$

The vectors ω^i are self-dual 2-forms living in the lattice $H^2(K3,\mathbb{R}) = \mathbb{R}^{3,19}$, and describe the moduli space of Einstein metrics of unit volume. The space of inequivalent compactifications is a discrete quotient of the Grassmannian $O(4,20)/O(4)\times O(20)$:

$$O(4,20;\mathbb{Z})\backslash O(4,20)/O(4)\times O(20), \tag{2.2}$$

since physical quantities depend on a choice of $(0, 2, \text{or } 4)$-brane charge in the lattice $\Gamma^{(4,20)} \subset \mathbb{R}^{(4,20)}$ and a choice of background. The discrete quotient is by symmetries of the lattice. In the absence of B-fields we can invert the volume of the $K3$ by a transformation which involves permuting the first two entries of the vectors [1]. The usual T-duality inverting the volume of the torus can also be described in a similar language with $\Gamma^{(4,4)} \subset \mathbb{R}^{(4,4)}$ replacing $\Gamma^{(4,20)} \subset \mathbb{R}^{(4,20)}$. These dualities are called Fourier-Mukai dualities in view of their action on the gauge theory describing the dual brane systems. A different element of the T-duality group inverts the volume of $K3$ in the presence of special B-fields present in the perturbative orbifold limit of $K3$ [2]–[5]. The case $B = 0$ will be of interest here.

2.1 Zero-brane on $\mathbb{T}^4$

We will consider the consequences of this duality on a system of zero-brane on $\mathbb{T}^4$ with $B = 0$. The moduli space of a zero-brane on $\mathbb{T}^4$ in the large volume limit can safely be said to be identical to the $\mathbb{T}^4$ itself. All the Kähler and complex structure parameters of the moduli space of the zero-brane are identical to those of the base space itself.

Let us denote by $M_{(Q_4,Q_0)}(X)$ the moduli space of Q_0 zero-branes and Q_4 4-branes on X, where X is $\mathbb{T}4$, a four-torus, or $K3$. The system (Q_4, Q_0) is associated with the Mukai vector $(Q_4, Q_0 - Q_4)$ in the case of $K3$, and (Q_4, Q_0) in the case of $\mathbb{T}^4$ [6].

The quantity of immediate interest will be the volume of the moduli space of a zero-brane on $\mathbb{T}^4$, which we will denote as $\mathrm{Vol}(M_{0,1}(T(V)))$. In the large volume limit, by the above reasoning $\mathrm{Vol}(M_{0,1}(T(V))) = V$.

Now consider varying the geometry of the torus, reducing its volume while keeping all other Kähler and complex structure parameters fixed. Once we reach the small volume region, we can use T-duality to map to the large volume region, and

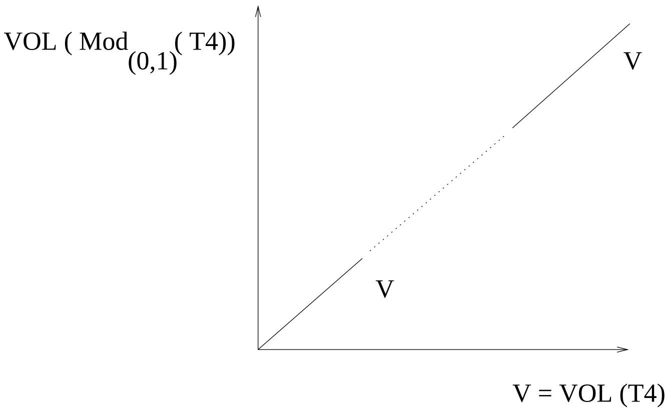

Figure 1: Volume of moduli space of zero-brane as a function of volume of base T^4.

at the same time map the zero-brane to 4-brane. By considering the moduli space of the 4-brane in the large volume limit we can learn about the moduli space of the zero-brane in the small volume limit.

The 4-brane in the large volume limit is described by U(1) gauge theory. The moduli space is the space of flat connections on the torus. Since the moduli space of flat connections on the large torus is the dual torus which has small volume, we deduce that the moduli space of the zero brane as a function of V behaves as $\mathrm{Vol}(M_{1,0}(T(1/V))) = V$ in the region of small V,

To summarize:

$$\mathrm{Vol}(M_{0,1}(T(V))) = V \text{ for large V }, \qquad \mathrm{Vol}(M_{0,1}(T(V))) = V \text{ for small V }. \tag{2.3}$$

The simplest way to interpolate between these two limits is to take $\mathrm{Vol}(M_{0,1}(T(V))) = V$ for all V. According to this guess the volume is equal to 1 at the self-dual point. This is illustrated in the figure. Also note that the T-duality implies that the Kähler parameters obey:

$$\omega^i(M_{0,1}(T(V, \omega^i)) = \omega^i \tag{2.4}$$

in both the small and large volume limits.

2.2 Zero-brane on $K3$

Now apply the same considerations to a zero-brane on $K3$ with $B = 0$. We certainly have $\mathrm{Vol}(M_{0,1}(K(V))) = V$ in the large volume limit. Now consider the small volume limit. The T-duality gives us a $K3$ of large volume, and maps the zero-brane to a system of 4-brane and zero-brane. This is because Fourier-Mukai duality acts simply on the Mukai vector which includes the contribution from the curvature of the $K3$. U(1) gauge theory with an instanton describes one 4-brane with no total zero-brane charge, because the charge of the U(1) instanton cancels the charge induced from the curvature due to the term $\int C^{(1)} \wedge R \wedge R$ in the 4-brane action.

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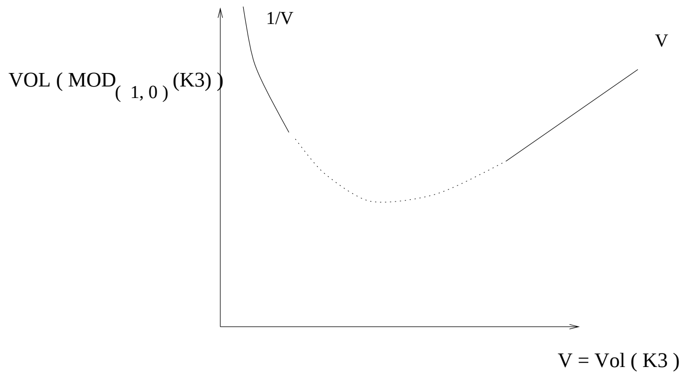

Figure 2: Volume of moduli space of zero-brane as a function of volume of base K3.

Now the moduli space of U(1) gauge theory with a single instanton on $K3$ is just $K3$ with geometry identical to the base space: $\mathrm{Mod}_{1,0}(K3) = K3$. We can see this by the Polchinski D-brane construction of such a system. We can start with the zero-brane and 4-brane being separated by a short distance in directions transverse to the $K3$, and take a limit as the zero-brane approaches the 4-brane. An open string end-point can end on a zero-brane or a 4-brane. The worldsheet CFT description remains valid, and if g_s is taken to be small, tree level CFT gives the correct description. From this description a string end-point with Dirichlet boundary conditions along the $K3$ directions can end at any position on the $K3$, which is the location of the zero-brane. An open string connecting zero-brane and 4-brane will have a family of supersymmetric configurations parametrized by the $K3$. The moduli space of such boundary CFTs will clearly the contain the $K3$ itself. There will be boundary marginal operators in this CFT which change the location of the zero-brane, and their two-point functions are determined by the metric on the base space. These two-point functions in turn define the metric on the moduli space of the boundary CFTs. After we T-dualize a small volume $K3$ of volume V to a $K3$ of large volume $1/V$ we can use the above argument to show that $\mathrm{Vol}(M_{1,0}(K(1/V))) = 1/V$ in the large volume limit. This implies that $\mathrm{Vol}(M_{0,1}(K(V))) = 1/V$.

To summarize,

$$\mathrm{Vol}(M_{0,1}(K(V))) = V \text{ for large V}\,, \qquad \mathrm{Vol}(M_{0,1}(K(V))) = \frac{1}{V} \text{ for small V}\,. \tag{2.5}$$

This is illustrated in the figure. We are further inclined to guess that the volume has one minimum at $V = 1$, where $\mathrm{Vol}(M_{0,1}(K(V))) = 1$. We also deduce from the T-duality that the complex structure and Kähler parameters of $M_{0,1}(K(V))$ are the same in the small volume limit as in the large volume limit.

2.3 Extension to N zero-branes

One obvious extension is to discuss N zero-branes. We have $S^N(X)$ as the moduli space in the large volume limit for N zero-branes. In the limit of small $K3$ we use the duality to map to a system described by U(N) gauge theory with N instantons on a large dual $K3$. The instantons are certainly not ordinary stable sheaves in this case since the Mukai dimension formula gives zero. Presumably it can be proved that they are necessarily point-like. So we have a symmetric product of the dual $K3$, which is a $K3$ of large volume. So the moduli space of N zero-branes interpolates between being a symmetric product of N $K3$'s of volume V in the large volume limit and being a symmetric product of N $K3$'s of volume $1/V$ in the small volume limit.

2.4 Note on boundary state definition of the quantum volumes.

We emphasize that the above arguments have used duality to deduce the properties of the quantum volume of the moduli space, a quantity which is independently defined in terms of boundary states in the CFTs describing the compactifications. We are not using the T-duality to define the quantum volumes.

We can take the coordinates on the end-point of the string ending on the zero-brane and consider two-point functions

$$\langle x^i(\tau)x^j(\tau')\rangle = -\alpha' g^{ij}\log(\tau-\tau')^2\,. \tag{2.6}$$

We can express this correlation function alternatively as:

$$\frac{\partial}{\partial\phi^i}\frac{\partial}{\partial\phi^j}\int dg dX e^{\int S+\int\phi_i\partial_n X^i}\,. \tag{2.7}$$

By differentiating we bring down boundary operators. We can express this in terms of boundary states

$$\langle 0|B(\phi)\rangle \tag{2.8}$$

by taking two derivatives. Defined in this way we can extend the definition of the metric to abstract CFT's and obtain a definition of the quantum volume. The relevant two-point functions analogous to (2.6) will have to be computed from more explicit knowledge about the CFT and its boundary marginal operators. The T-duality argument gives a prediction for this quantum volume.

3. Discussion

We showed how some elementary facts about simple degenerate instanton moduli spaces gives concrete information about quantum volumes. These degenerate instanton moduli spaces appear as sub-strata of larger instanton moduli spaces. A lot of information about such stratifications and their symmetries (acting on the instanton numbers and magnetic fluxes characterizing the strata) are present in BPS mass formulae and associated BPS splittings of the kind studied in detail in [7] for the case of $\mathbb{T}^4$. The interpretation of such symmetries in the case of $K3$ has to take into account the α' corrections to instanton moduli spaces of the kind discussed here. Information about less degenerate smooth strata, e.g. of the kind in [8] could also be used in conjunction with duality to obtain information about quantum corrected geometries.

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The emergence of a classical geometry from string-corrected moduli spaces exhibited in figure 2. apears somewhat remarkable. It is tempting to speculate that some simple rules in string theory dictate the appearance of the large volume $K3$. For example one might suspect that there is a bound on the volume seen by any probe as one moves in moduli space. Alternatively there might be some constraints on the products of the volumes seen by different probes. To make these speculations more precise would require integrating different candidate definitions of quantum volumes of cycles, e.g. those considered in [9] and refs. therein. Subtleties of the kind discussed in [10] may have to be dealt with.

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